Exercises LA 2025/07/21

Ex. 1

Prove that there does not exist an integer x such that

$$x + x = 1.$$

Ex. 2

Recall the following definition of an equivalence (relation).

Definition 1 (Equivalence/Relation)**.** *Let X be a set. An equivalence (relation) on X is a partition on X, i.e., there exists a collection of subsets $\mathcal{S}_\alpha$, $\alpha \in I$, where I is some index set, such that, $\forall x \in X$, there exists a unique $\alpha \in I$, such that $x \in \mathcal{S}_\alpha$. And we call each such subset $\mathcal{S}_\alpha$ an equivalent class.*

1. Prove that the collection of subsets $[a, b]$ defines an equivalence on $\widetilde{\mathbb{Z}^2}$, where we recall

   $$\widetilde{\mathbb{Z}^2} := \left\{ (a, b) \in \mathbb{Z}^2 \mid b \neq 0 \right\}.$$

   and for $(a, b) \in \widetilde{\mathbb{Z}^2}$,

   $$[a, b] := \left\{ (x, y) \in \widetilde{\mathbb{Z}^2} \mid ay = bx \right\}.$$

2. (Optional)

   For $x, y \in X$, write $x \sim y$ if $x, y \in \mathcal{S}_\alpha$. Prove that for all $x, y, z \in X$:

   (T) $x \sim y$ and $y \sim z \implies x \sim z$ (transitive);

   (S) $x \sim y \implies y \sim x$ (symmetric);

   (R) $x \sim x$ (reflexive).