Intersection numbers and the counting of lattice points

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Abstract. Originating from the study of the intersection of two plane curves, intersection theory has developed into a prominent field in modern algebraic geometry. The intersection theory of the moduli space of curves was initiated by Mumford in the 1980s and attracted a large amount of attention due to Witten's discovery of its connections to integrable systems, as well as its many applications in string theory and enumerative geometry.

The main subject of this paper is the expression of descendent integrals on moduli spaces of curves as lattice point counts of a polytope. This relation was first established by Afandi through Ehrhart theory in discrete geometry, as well as a kind of polynomiality property of descendent integrals due to Liu-Xu.

Our work strengthens Afandi's theorem by dropping a genus shift in the assertion and, at the same time, by presenting a more succinct statement. Our proof is an induction using the DVV formula. The main technical difficulty lies in proving an inequality by using the Leibniz rule for finite differences and using Eynard-Orantin theory to show the positivity of normalized 3-point functions.

Contents

Key words and phrases. Moduli space of curves, intersection number, Ehrhart polynomial.

1. Introduction

The roots of intersection theory arise from the classical problem of determining the intersection of two plane curves, or more generally, of multiple algebraic hypersurfaces in n-dimensional space. Isaac Newton investigated this problem in his *Principia*. It was later formulated as the celebrated Bézout's theorem.

Theorem 1.1 (Bézout's theorem)**.** *Let C_1 and C_2 be two projective plane curves of degrees m and n, defined over an algebraically closed field F. Then they intersect at exactly mn points, counting multiplicities.*

Plücker's notion of the *class* of a curve was an application of Bézout's theorem. The class of a plane curve C is defined as the number of tangents to C through a point Q. Plücker gave an explicit formula for the class of plane curves. Let $F(x, y, z)$ be the homogeneous polynomial defining C, pick $Q = (a : b : c)$. Define the *polar curve* C_Q by:

$$F_Q(x, y, z) = a\frac{\partial F}{\partial x} + b\frac{\partial F}{\partial y} + c\frac{\partial F}{\partial z}$$

This definition ensures that a nonsingular point on C is on C_Q if and only if the tangent line to C at that point goes through Q. On the other hand, since partial derivatives vanish at singular points, all singular points of C are on C_Q. The number of points where C and C_Q intersect, which can be expressed in terms of degree using Bézout's theorem, is a combination of the class of C and the singular points of C, explicitly

$$\#C \cap C_Q = \deg C \deg C_Q = n(n-1) = \text{class}(C) + \#\text{singular points}$$

where n is the degree of C.

Plücker's first formula reveals that ordinary nodes contribute 2 intersection points (with intersection multiplicity 2), and ordinary cusps contribute 3 points (with intersection multiplicity 3).

$$n(n-1) = \text{class}(C) + 2\delta + 3\kappa$$

where n is the degree of C, δ is the number of ordinary nodes and κ is the number of ordinary cusps.

In 1847 Salmon obtained an analogous formula for surfaces. Let $S \subset \mathbb{P}^3$ be a surface, then the degree of the dual surface $S^\vee$ (now called the *second class*) is the number of points P on S such that the tangent plane at P contains a general line l. However, different from the case of plane curves, there is the problem of excess intersection when analyzing contributions of singular points. Specifically, when the surface S is singular along a curve C, Salmon calculated the contributions when C is a line, a double line and general curves.

On the other hand, Chasles, de Jonquières and many mathematicians avoided the issue of excess intersections by calculating intersections only on certain special spaces.

Later, Severi developed a procedure for calculating intersection multiplicity, which was corrected and completed by Lazarsfeld and Macaulay.

The moduli space of curves was first studied by Riemann. Nowadays, it lies at the center of the confluence of algebraic geometry, number theory and mathematical physics. In the past few decades the subject has also gained importance with input from string theory.

In this paper, we will touch upon only one aspect of moduli space of curves – intersection theory – and explore its connection with Ehrhart theory.

In the early 1990s, Witten's conjecture [17], first solved by Kontsevich [11], invigorated the study of intersection theory on moduli spaces by connecting it with integrable systems. The Witten-Kontsevich theorem enabled intersection numbers involving ψ classes, or descendant integrals, to be calculated via a recursion formula. This is the starting point of modern enumerative geometry or Gromov-Witten theory, which culminated in the solution of the long-standing enumeration problem of rational curves on Calabi-Yau manifolds, by Lian-Liu-Yau [12] and Givental [8] independently.

Recently, Afandi [1] established a very interesting connection between intersection numbers on the moduli space of curves and Ehrhart theory – the polynomiality occurs simultaneously in intersection theory and counting lattice points in a polytope. For more related works, see [9, 10, 14, 18].

In this paper, we briefly review the Witten-Kontsevich theorem as well as Ehrhart theory. After that, we prove a strengthening of Afandi's theorem [1] in Sections 4-6.

2. Intersection numbers and the Witten-Kontsevich theorem

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable n-pointed genus g complex algebraic curves. Denote by π the morphism that forgets the last marked point

$$\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

Denote by $\sigma_1, \ldots, \sigma_n$ the canonical sections of π. Let ω_π be the relative dualizing sheaf. There are three families of tautological classes on $\overline{\mathcal{M}}_{g,n}$.

$$\begin{aligned}
\psi_i &= c_1(\sigma_i^*(\omega_\pi)), \quad 1 \le i \le n \\
\kappa_i &= \pi_*(\psi_{n+1}^{i+1}) \\
\lambda_k &= c_k(\mathbb{E}), \quad 1 \le k \le g,
\end{aligned}$$

where $\mathbb{E} = \pi_*(\omega_\pi)$ is called the Hodge bundle.

Intuitively, ψ_i is the first Chern class of the line bundle whose fiber is the cotangent space of the curve at the i-th marked point and the fiber of $\mathbb{E}$ is the space of holomorphic one-forms on the algebraic curve.

We adopt Witten's notation for intersection numbers:

$$\langle \tau_{d_1} \cdots \tau_{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \mid \lambda_1^{k_1} \cdots \lambda_g^{k_g} \rangle := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \lambda_1^{k_1} \cdots \lambda_g^{k_g}.$$

These are also called Hodge integrals, which are rational numbers, and their total degrees should add up to $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$.

Intersection numbers of pure ψ classes $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ are often called descendent integrals. Intersection numbers of pure κ classes $\langle \kappa_{a_1} \cdots \kappa_{a_m} \rangle$ are called higher Weil-Petersson volumes. Integrals of κ_1 class $\langle \kappa_1^{3g-3+n} \rangle_{g,n}$ are the classical Weil-Petersson volumes.

The κ classes on $\overline{\mathcal{M}}_g$ was first introduced by Mumford [16] and their generalization to $\overline{\mathcal{M}}_{g,n}$ was due to Arbarello-Cornalba [2].

2.1. **Witten-Kontsevich theorem.** In 1990, Witten [17] conjectured that the generating function of descendent integrals is governed by the KdV hierarchy. Witten's conjecture was first proved by Kontsevich [11]. Kontsevich's proof used a novel combinatorial description of moduli spaces and Feynman diagram techniques. Now we have many different proofs of the Witten-Kontsevich theorem due to Chen-Li-Liu, Kazarian-Lando, Kim-Liu, Okounkov-Pandharipande, Mirzakhani.

Witten's motivation comes from two seemingly unrelated mathematical models that both describe the physical theory of two-dimensional gravity: the counting of triangulations of surfaces via matrix integrals and the intersection theory of $\overline{\mathcal{M}}_{g,n}$. The partition function of the first model is known to obey the KdV hierarchy.

The KdV hierarchy is a family of differential equations labeled by $n \geq 1$,

$$\frac{\partial U}{\partial t_n} = \frac{\partial R_{n+1}}{\partial t_0},$$

where R_n are Gelfand-Dikii differential polynomials in $U, \partial U/\partial t_0, \partial^2 U/\partial t_0^2, \ldots$, defined recursively by

$$R_1 = U, \qquad \frac{\partial R_{n+1}}{\partial t_0} = \frac{1}{2n+1}\left(\frac{\partial U}{\partial t_0} R_n + 2U\frac{\partial R_n}{\partial t_0} + \frac{1}{4}\frac{\partial^3 R_n}{\partial t_0^3}\right).$$

The first few terms are given by

$$R_2 = \frac{1}{2}U^2 + \frac{1}{12}\frac{\partial^2 U}{\partial t_0^2},$$

$$R_3 = \frac{1}{6}U^3 + \frac{U}{12}\frac{\partial^3 U}{\partial t_0^3} + \frac{1}{24}(\frac{\partial U}{\partial t_0})^2 + \frac{1}{240}\frac{\partial^4 U}{\partial t_0^4},$$

$$\vdots$$

The Witten-Kontsevich theorem asserts that the generating function

$$F(t_0, t_1, \ldots) = \sum_g \sum_{\mathbf{n}} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!} \tag{1}$$

is a τ-function for the KdV hierarchy, i.e. $U = \partial^2 F/\partial t_0^2$ is a solution to all equations in the KdV hierarchy. The first equation in the KdV hierarchy is the classical KdV equation

$$\frac{\partial U}{\partial t_1} = U\frac{\partial U}{\partial t_0} + \frac{1}{12}\frac{\partial^3 U}{\partial t_0^3}.$$

In addition, F obey the following string and dilaton equations

$$\frac{\partial F}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1}\frac{\partial F}{\partial t_i}$$

$$\frac{\partial F}{\partial t_1} = \frac{1}{24} + \sum_{i=0}^{\infty} \frac{2i+1}{3} t_i \frac{\partial F}{\partial t_i}.$$

2.2. **Virasoro constraints.** The Witten-Kontsevich theorem has an important reformulation in terms of the Virasoro constraints.

Define a family of differential operators L_k for $k \geq -1$ by

$$L_k = -\frac{1}{2}(2k+3)!!\frac{\partial}{\partial t_{k+1}} + \frac{1}{2}\sum_{j=0}^{\infty}\frac{(2(j+k)+1)!!}{(2j-1)!!}t_j\frac{\partial}{\partial t_{j+k}}$$
$$+\frac{1}{4}\sum_{d_1+d_2=k-1}(2d_1+1)!!(2d_2+1)!!\frac{\partial^2}{\partial t_{d_1}\partial t_{d_2}} + \frac{\delta_{k,-1}t_0^2}{4} + \frac{\delta_{k,0}}{48},$$

These operators satisfy the Virasoro relations

$$[L_n, L_m] = (n-m)L_{n+m}.$$

Dijkgraaf, Verlinde, and Verlinde [4] proved that the KdV form of Witten's conjecture is equivalent to the following Virasoro constraints.

Proposition 2.1. *(DVV formula) Let F be the generating function of descendent integrals defined in* (1)*. We have $L_k(\exp F) = 0$ for $k \geq -1$. More explicitly,*

$$\begin{aligned}\langle\tau_{k+1}\tau_{d_1}\cdots\tau_{d_n}\rangle_g = \frac{1}{(2k+3)!!}\Bigg[&\sum_{j=1}^n \frac{(2k+2d_j+1)!!}{(2d_j-1)!!}\langle\tau_{d_1}\cdots\tau_{d_j+k}\cdots\tau_{d_n}\rangle_g \\ &+\frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\tau_{d_1}\cdots\tau_{d_n}\rangle_{g-1} \\ &+\frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\sum_{\underline{n}=I\coprod J}\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\Bigg].\end{aligned}$$

The special cases of the DVV formula when $k=-1$ and $k=0$ are just the string and the dilaton equation respectively.

$$\textit{The string equation: } \langle\tau_0\tau_{d_1}\ldots\tau_{d_n}\rangle_g = \sum_{i=1}^n \langle\tau_{d_1}\ldots\tau_{d_i-1}\ldots\tau_{d_n}\rangle_g$$

$$\textit{The dilaton equation: } \langle\tau_1\tau_{d_1}\ldots\tau_{d_n}\rangle_g = (2g-2+n)\langle\tau_{d_1}\ldots\tau_{d_n}\rangle_g$$

Here is the closed formula for one-point intersection numbers:

$$\langle\tau_{3g-2}\rangle_g = \frac{1}{24^g g!}$$

When $g=0$, there is the well-known identity:

$$\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_0 = \binom{n-3}{d_1,\ldots,d_n}.$$

Example 2.2. $\langle\tau_1\rangle_1 = \dfrac{1}{24}$

Example 2.3. $\langle\tau_2\tau_3\tau_2\tau_0\rangle_g = \langle\tau_1\tau_3\tau_2\rangle_g + \langle\tau_2\tau_2\tau_2\rangle_g + \langle\tau_2\tau_3\tau_1\rangle_g = \langle\tau_2\tau_2\tau_2\rangle_g + 4g\langle\tau_2\tau_3\rangle_g$

Example 2.4.

$$\begin{aligned}\langle\tau_4\tau_5\tau_0^2\rangle_g &= \langle\tau_3\tau_5\tau_0\rangle_g + \langle\tau_4\tau_4\tau_0\rangle_g \\ &= \langle\tau_2\tau_5\rangle_g + \langle\tau_3\tau_4\rangle_g + 2\langle\tau_3\tau_4\rangle_g\end{aligned}$$

2.3. **Hodge Integrals and Faber's algorithm.** Hodge integrals are intersection numbers involving ψ, κ and λ classes on $\overline{\mathcal{M}}_{g,n}$.

Faber's algorithm [7] reduces the calculation of general Hodge integrals to those with pure ψ classes.

The ELSV formula [5] relates single Hurwitz numbers to Hodge integrals.

Theorem 2.5 (ELSV formula)**.** *Let $n = l(\mu)$ and $r = 2g-2+|\mu|+n$. Then*

$$H_{g,\mu} = r!\prod_{i=1}^n\left(\frac{\mu_i^{\mu_i}}{\mu_i!}\right)\int_{\overline{\mathcal{M}}_{g,n}}\frac{1-\lambda_1+\cdots+(-1)^g\lambda_g}{(1-\mu_1\psi_1)\cdots(1-\mu_n\psi_n)},$$

The ELSV formula, as well as many Hodge integral identities can be derived from the Mariño-Vafa formula [13].

As is well-known to experts, intersection numbers involving mixed ψ and κ classes can be reduced to integrals of pure κ classes or pure ψ classes by using the projection formula. For example, the following formula reducing ψ class one at a time can be found in [15].

Proposition 2.6. *Let $d_n \geq 1$, $\mathbf{b} = (b_1, b_2, \dots) \in N^\infty$ where N^∞ is the semigroup of sequences with nonnegative integers b_i and $b_i = 0$ for sufficiently large i. Define*

$$\kappa(\mathbf{b}) \triangleq \prod_{i\geq 1} \kappa_i^{b_i}$$

Then

$$\langle \tau_{d_1} \cdots \tau_{d_n} \kappa(\mathbf{b}) \rangle_g = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{d_1} \cdots \tau_{d_{n-1}} \kappa(\mathbf{L}') \kappa_{|\mathbf{L}|+d_n-1} \rangle_g.$$

2.4. **An integer-valued polynomial.** In [14], Liu-Xu discovered genus polynomiality of intersection numbers.

Theorem 2.7 ([14, Theorem 4.1])**.** *For any fixed set $\mathbf{d} = (d_1, \dots, d_n)$ of nonnegative integers with $|\mathbf{d}| := d_1 + \cdots + d_n$, the following function*

$$P_{d_1,\dots,d_n}(g) = \frac{\langle \tau_{d_1} \cdots \tau_{d_n} \tau_{3g-2+n-|\mathbf{d}|} \rangle_g}{\langle \tau_{3g-2} \rangle_g} \prod_{i=1}^{n} (2d_i+1)!! \tag{2}$$

is a polynomial in g with highest-degree term $6^{|\mathbf{d}|} g^{|\mathbf{d}|}$. Moreover, $P_{d_1,\dots,d_n}(g)$ is integer-valued, i.e., $P_{d_1,\dots,d_n}(g) \in \mathbb{Z}$ whenever $g \in \mathbb{Z}$.

These polynomials $P_{d_1,\dots,d_n}(g)$ are determined uniquely by the initial values $P_\emptyset(g) = P_{0,\dots,0}(g) = 1$ and the recursive relation

$$\begin{aligned} P_{d_1,\dots,d_n}(g) &= \sum_{j=2}^{n} (2d_j+1) P_{d_2,\dots,d_j+d_1-1,\dots,d_n}(g) \\ &+ \prod_{j=1}^{d_1} (6g+2n-2|\mathbf{d}|+2j-5) P_{d_2,\dots,d_n}(g) + 12g \sum_{r+s=d_1-2} P_{r,s,d_2,\dots,d_n}(g-1) \\ &+ \sum_{\substack{r+s=d_1-2 \\ I \coprod J=\{2,\cdots,n\}}} 24^{g'} \langle \tau_r \prod_{i\in I} \tau_{d_i} \rangle_{g'}^{\mathbf{w}} \prod_{j=1}^{g'} (g+1-j) P_{s,d_J}(g-g'), \end{aligned} \tag{3}$$

where in the last term we used the normalized tau function

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g^{\mathbf{w}} := \prod_{i=1}^{n} (2d_i+1)!! \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g.$$

3. Ehrhart theory and Afandi's work

3.1. **Ehrhart theory.** For background and important theorems in Ehrhart theory, we follow the book [3].

Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex d-polytope with vertices $v_1, \dots, v_n \in \mathbb{Z}^d$, then

$$\mathcal{P} = \text{Conv}(v_1, \dots, v_n) \subset \mathbb{R}^d.$$

For $t \in \mathbb{Z}_{>0}$, $t\mathcal{P}$ denotes the t^{th} dilate of the polytope $\mathcal{P} \subset \mathbb{R}^d$

$$t\mathcal{P} = \text{Conv}(tv_1, \dots, tv_n) = \{tp : p \in \mathcal{P}\}.$$

The lattice-point enumerator for the t^{th} dilate of $\mathcal{P} \subset \mathbb{R}^d$ is defined as

$$L_{\mathcal{P}}(t) := \#(t\mathcal{P} \cap \mathbb{Z}^d).$$

The fundamental theorem concerning lattice-point enumeration, named in honor of Eugène Ehrhart, is the following.

Theorem 3.1 (Ehrhart's Theorem)**.** *If $\mathcal{P}$ is an integral convex d-polytope, then $L_{\mathcal{P}}(t)$ is a rational polynomial in t of degree d.*

$L_{\mathcal{P}}(t)$ is called the Ehrhart polynomial of $\mathcal{P}$.

Following [1], we define the notion of polytopal complexes and the f^*-vector of an integral d-polytope.

Definition 3.2. A *polytopal complex* $\mathcal{K}$ is a finite collection of polytopes in $\mathbb{R}^d$ satisfying the following three properties:

(1) the empty polytope is in $\mathcal{K}$,
(2) if $\mathcal{P} \in \mathcal{K}$, then all the faces f of $\mathcal{P}$ are also in $\mathcal{K}$,
(3) the intersection $\mathcal{P} \cap \mathcal{Q}$ of two polytopes $\mathcal{P}, \mathcal{Q} \in \mathcal{K}$ is a face of both $\mathcal{P}$ and $\mathcal{Q}$.

The elements of $\mathcal{K}$ are called the *faces* of $\mathcal{K}$. The *dimension* of $\mathcal{K}$ is the largest dimension of the faces of $\mathcal{K}$.

Definition 3.3. Let $\mathcal{P}$ be an integral convex d-polytope. The *f^*-vector* of $\mathcal{P}$ is the unique integer tuple $(f_0^*, \dots, f_d^*) \in \mathbb{Z}^{d+1}$ such that

$$L_{\mathcal{P}}(t) = \sum_{k=0}^{d} f_k^* \binom{t-1}{k}$$

A generalization of the integral d-polytope is the *open d-polytope*; it is the relative interior of an integral d-polytope. We also generalize polytopal complexes.

Definition 3.4. An *integral partial polytopal complex* $\mathcal{K}$ is a finite disjoint union of open integral polytopes. The elements of $\mathcal{K}$ are called the faces of $\mathcal{K}$. The dimension of $\mathcal{K}$ is the largest dimension of the faces of $\mathcal{K}$. The Ehrhart polynomial of $\mathcal{K}$, denoted $L_{\mathcal{K}}(g)$, is the sum of the Ehrhart polynomials of each face of $\mathcal{K}$.

Then it makes sense to talk about the f^*-vector of an integral partial polytopal complex.

Theorem 3.5. *For a partial polytopal complex $\mathcal{K}$ of dimension d, we have the decomposition of its Ehrhart polynomial*

$$L_{\mathcal{K}}(t) = \sum_{i=0}^{d} f_i^* \binom{t-1}{i}$$

we call $(f_0^*, \dots, f_d^*)$ the f^*-vector of $\mathcal{K}$.

These tools are sufficient for us to introduce the following central theorem due to Breuer.

Theorem 3.6 (Breuer)**.** *An integer-valued polynomial $P(g)$ of degree d is the Ehrhart polynomial of an integral partial polytopal complex if and only if the f^*-vector $(f_0^*, \dots, f_d^*)$ of $P(g)$ is nonnegative i.e. $f_i^* \geq 0$ for all $0 \leq i \leq d$.*

For simplicity, if an integer-valued polynomial $P(g)$ satisfies the condition of the above theorem, we will call it an Ehrhart polynomial.

3.2. **Afandi's theorem.** The main theorem proved by Afandi is the following.

Theorem 3.7 (Afandi [1])**.** *For any fixed set $\mathbf{d} = (d_1, \cdots, d_n)$ of nonnegative integers with $n \geq 1$. Define*

$$C(\mathbf{d}) := \prod_{i=1}^{n} (2d_i + 1)!!,$$
$$m(\mathbf{d}) = m := \left\lceil \frac{2 - n + |\mathbf{d}|}{3} \right\rceil - 1.$$

Then there exists an integral partial polytopal complex $\mathcal{P}_{\mathbf{d}}$ with dimension $|\mathbf{d}|$ and volume $vol(\mathcal{P}_{\mathbf{d}}) = 6^{|\mathbf{d}|}$ such that

$$24^{g+m}(g+m)!C(\mathbf{d}) \int_{\overline{\mathcal{M}}_{g+m,n+1}} \psi_1^{d_1} \dots \psi_n^{d_n} \psi_{n+1}^{3(g+m)-2+n-|\mathbf{d}|} = \#\{\text{integer lattice points in } g\mathcal{P}_{\mathbf{d}}\}$$

where $g\mathcal{P}_{\mathbf{d}}$ is the g^{th} dilate of $\mathcal{P}_{\mathbf{d}}$.

Afandi's theorem can be rephrased as that $P_{d_1,\cdots,d_n}(g+m)$, defined in (2), is an Ehrhart polynomial. The shift $m(\mathbf{d})$ is indispensable in Afandi's inductive proof using the DVV formula. One may naturally ask whether this shift of genus could be dropped in Afandi's theorem. The answer is affirmative. This is what we are going to show in the next section.

Note that we have the string and dilaton equations for $P_{d_1,\dots,d_n}(g)$ when $d_1 = 0$ or 1.

$$P_{0,d_2,\dots,d_n}(g) = \sum_{j=2}^{n} (2d_j + 1) P_{d_2,\dots,d_j-1,\dots,d_n}(g) + P_{d_2,\dots,d_n}(g),$$
$$P_{1,d_2,\dots,d_n}(g) = (6g + 3n - 6) P_{d_2,\dots,d_n}(g).$$

If we assume $d_i \geq 1$ for all $1 \leq i \leq n$, then $m(\mathbf{d}) \geq 0$. Note that if $f(g)$ is an Ehrhart polynomial, then so is $f(g+1)$. Therefore dropping the shift $m(\mathbf{d})$ is a strengthening of Afandi's theorem.

4. Improvement of Afandi's theorem

For the polynomial $P_{d_1,\cdots,d_n}(g)$, we will treat the cases $n = 1$ and $n \geq 2$ separately.

The following formula of $P_d(g)$ was derived from 2-point function due to Dijkgraaf.

Lemma 4.1 ([14, Corollary 4.5])**.** *Let $d \geq 0$ be a nonnegative integer. Then*

$$\frac{P_d(g)}{(2d+1)!!} = \sum_{i=0}^{\lfloor \frac{d-1}{3} \rfloor} \sum_{k} \frac{12^k k!(k+i)!}{i!(2k+1)!} \binom{k-1}{d-3i-k} \binom{g}{k+i} + (-1)^{d \bmod 3} \binom{g-1}{\lfloor \frac{d}{3} \rfloor}, \tag{4}$$

where the summation range of k is $\max(\lceil \frac{d-3i+1}{2} \rceil, 1) \leq k \leq d - 3i$.

Theorem 4.2. *Let $d \geq 0$ be a nonnegative integer. Then $P_d(g)$ is an Ehrhart polynomial. Namely, if we write*

$$P_d(g) = \sum_{i=0}^{d} a_i \binom{g-1}{i},$$

then $a_i \geq 0$ for all $0 \leq i \leq d$.

Proof. From Lemma 4.1, the negative contribution comes from the last term of (4). So we only need to check that when $d = 3m+1$, we have $a_m \geq 0$. Note that the contribution to $\binom{g-1}{m}$ of the first term in the right-hand side of (4) only occurs when $i = m$ and $k = 1$, so we have

$$a_m = (6m+3)!! \left(\frac{12(m+1)!}{m! \cdot 6} - 1 \right) = (2m+1)(6m+3)!! > 0.$$

The statement is proven. □

Theorem 4.3. *For any fixed set $\mathbf{d} = (d_1, \cdots, d_n)$ of nonnegative integers with $n \geq 2$, the polynomial $P_{d_1,\cdots,d_n}(g-1)$ is an Ehrhart polynomial. Namely, if we write*

$$P_{d_1,\cdots,d_n}(g) = \sum_{k=0}^{|\mathbf{d}|} I_{d_1,\cdots,d_n}(k) \binom{g}{k}, \tag{5}$$

then $I_{d_1,\cdots,d_n}(k) \geq 0$ for all $0 \leq k \leq |\mathbf{d}|$.

From (3), we have a recursive formula for the coefficients $I_{d_1,\cdots,d_n}(k)$ (see [14, Page 44]).

$$\begin{aligned} I_{d_1,\ldots,d_n}(k) &= \sum_{j=2}^{n} (2d_j+1) I_{d_2,\ldots,d_j+d_1-1,\ldots,d_n}(k) \\ &\quad + \sum_{i=\max(0,k-d_1)}^{k} c_{k-i}(d_1, 2n-2|\mathbf{d}|+6i-5) \binom{k}{i} I_{d_2,\ldots,d_n}(i) \\ &\quad + 12k \sum_{r+s=d_1-2} I_{r,s,d_2,\ldots,d_n}(k-1) + \sum_{\substack{r+s=d_1-2 \\ I \coprod J = \{2,\cdots,n\}}} 24^{g'} \langle \tau_r \prod_{i\in I} \tau_{d_i} \rangle_{g'}^{\mathbf{w}} \frac{k!}{(k-g')!} I_{s,d_J}(k-g'), \end{aligned} \tag{6}$$

where $c_t(d_1, m)$, $0 \leq t \leq d_1$ are the coefficients of

$$\prod_{j=1}^{d_1} (6x+m+2j) = c_0 + c_1 x + c_2 \binom{x}{2} + \cdots + c_{d_1-1} \binom{x}{d_1-1} + c_{d_1} \binom{x}{d_1}.$$

In order to prove Theorem 4.3, we need some preparations.

Lemma 4.4 ([14, Corollary 4.2])**.** *Let $\mathbf{d} = (d_1, \cdots, d_n)$ with $d_i \geq 0$.*

(1) *A positive integer $k \geq 1$ is a root of $P_{d_1,\ldots,d_n}(g)$ if and only if $k < \frac{|\mathbf{d}|-n+2}{3}$.*
(2) *0 is a root of $P_{d_1,\ldots,d_n}(g)$ if and only if $2 \leq n \leq |\mathbf{d}|+1$.*

Lemma 4.5. *Denote by $I_{d_1,\cdots,d_n}(k)$ the coefficient of $\binom{g}{k}$ in the expansion of $P_{d_1,\cdots,d_n}(g)$ as in (5).*

(1) *For $d \geq 0$, we have $I_d(k) \geq 0$ when $k > \lfloor \frac{d}{3} \rfloor$*
(2) *For $n \geq 2$, we have $I_{d_1,\cdots,d_n}(k) = 0$ when $k < \lceil \frac{|\mathbf{d}|+2-n}{3} \rceil$*

Proof. The assertion of (1) follows from an inspection of the last term of (4) since $\binom{g}{i} = \binom{g-1}{i} + \binom{g-1}{i-1}$.

The assertion of (2) follows from Lemma 4.4. □

The proof of Theorem 4.3 consists of three steps.

Step 1. We show that Theorem 4.3 holds when $n = 2$, namely $I_{d_1,d_2}(k) \geq 0$.

Step 2. Assume $n \geq 3$, by Lemma 4.5, we may assume $k \geq \lceil \frac{|\mathbf{d}|+2-n}{3} \rceil$. Then apply the recursive formula (6) to inductively prove that $I_{d_1,\dots,d_n}(k) \geq 0$.

Step 3. In Step 2, we need the property of $c_{k-i}(d_1, 2n-2|\mathbf{d}|+6i-5) \geq 0$. From

$$c_{k-i}(d_1, 2n-2|\mathbf{d}|+6i-5) = \sum_{p=0}^{k-i} (-1)^{k-i-p} \binom{k-i}{p} \prod_{j=1}^{d_1} (6p+2n-2|\mathbf{d}|+6i-5+2j),$$

denote $b = 3i-3-(d_2+\cdots+d_n)+n$, we see that the factor

$$6p+2n-2|\mathbf{d}|+6i-5+2j = 6p+3-2(d_1+1-j)+2b.$$

Hence $c_{k-i}(d_1, 2n-2|\mathbf{d}|+6i-5) \geq 0$ is equivalent to Theorem 5.1, which will be proved in Section 5.

5. Proof of the key inequality

As mentioned, the following inequality was used in the proof of Theorem 4.3.

Theorem 5.1. *Let a, b, d be nonnegative integers satisfying $0 \leq a \leq d \leq 3a+b+1$. Then*

$$\sum_{p=0}^{a} (-1)^{a-p} \binom{a}{p} \prod_{j=1}^{d} (2b+3+6p-2j) \geq 0. \tag{7}$$

The above inequality has the following two equivalent formulations.

Theorem 5.2. *With the same condition as in Theorem 5.1,*

$$\sum_{n=0}^{d} S(n,a) 6^n e_{d-n}(2b+1, 2b-1, \dots, 2b+1-2(d-1)) \geq 0, \tag{8}$$

where e_k is the k-th elementary symmetric polynomial in its arguments, and $S(n,k)$ is the Stirling number of the second kind, which is the number of partitions of $\{1,\dots,n\}$ into k nonempty subsets.

Theorem 5.3. *With the same condition as in Theorem 5.1,*

$$\sum_{p=0}^{a} (-1)^{a-p} \binom{a}{p} (6p+2b+1)!! \prod_{j=1}^{3a-3p} (2b+1+6p-2d+2j) \geq 0. \tag{9}$$

Note that for fixed a and b the left-hand of (9) is a polynomial in d of degree $3a$.

The equivalences of the above three inequalities are not difficult to see. For example, the equivalence of (7) and (8) follow from the closed formula of Stirling numbers

$$S(n,a) = \frac{(-1)^a}{a!} \sum_{p=0}^{a} (-1)^p \binom{a}{p} p^n.$$

5.1. **Special cases.** We will give a proof of Theorem 5.1; however, first we will prove two special cases, which are needed in our inductive proof. In the following two lemmas, we prove inequality 9 (hence inequalities 7 and 8) for $d = 3a + b + 1$ and $d = 3a + b$.

Lemma 5.4. *Let a, b be nonnegative integers and $d = 3a + b + 1$, then*

$$\sum_{p=0}^{a}(-1)^{a-p}\binom{a}{p}(6p+2b+1)!!\prod_{j=1}^{3a-3p}(2b+1+6p-2d+2j) \geq 0. \tag{10}$$

Proof. (10) is equivalent to

$$\sum_{p=0}^{a}(-1)^{a-p}\binom{a}{p}(6p+2b+1)!!\prod_{j=1}^{3a-3p}(6p-6a-1+2j) \geq 0$$

$$\sum_{p=0}^{a}(-1)^{a-p}\binom{a}{p}(6p+2b+1)!!(-1)\cdot(-3)\cdots(6p-6a+1) \geq 0$$

$$\sum_{p=0}^{a}(-1)^{4a-4p}\binom{a}{p}(6p+2b+1)!!(6a-6p-1)!! \geq 0$$

$$\sum_{p=0}^{a}\binom{a}{p}(6p+2b+1)!!(6a-6p-1)!! \geq 0$$

The first step is because the product term $(6p - 6a - 1 + 2j)$ is negative at $j = 3a - 3p$, and thus a negative factor can be extracted from all terms in the product; the factor $(-1)^{3a-3p}$ combines with $(-1)^{a-p}$ to ensure nonnegativity. □

Lemma 5.5.

$$(6a+2b+1)!! - \sum_{p=0}^{a-1}\binom{a}{p}(6p+2b+1)!!(6a-6p-3)!! \geq 0 \tag{11}$$

where a, b are nonnegative integers.

Proof. For $n \geq 0$, we have the well-known identity

$$(2n+1)!! = \frac{2^{n+1}}{\sqrt{\pi}}\Gamma(n+\frac{3}{2}).$$

Hence

$$(6a+2b+1)!! = \frac{2^{3a+b+1}}{\sqrt{\pi}}\Gamma(3a+b+\frac{3}{2})$$

$$(6p+2b+1)!!(6a-6p-3)!! = \frac{2^{3a+b}}{\pi}\Gamma(3p+b+\frac{3}{2})\Gamma(3a-3p-\frac{1}{2})$$

And by

$$\Gamma(u)\Gamma(v) = \int_0^\infty\int_0^\infty x^{u-1}y^{v-1}e^{-(x+y)}dxdy,$$

we can rewrite the summation in (11) as follows:

$$S := \sum_{p=0}^{a-1}\binom{a}{p}(6p+2b+1)!!(6a-6p-3)!!$$

$$= \sum_{p=0}^{a-1} \binom{a}{p} \frac{2^{3a+b}}{\pi} \Gamma(3p + b + \frac{3}{2})\Gamma(3a - 3p - \frac{1}{2})$$

$$= \sum_{p=0}^{a-1} \binom{a}{p} \frac{2^{3a+b}}{\pi} \int_0^\infty \int_0^\infty x^{3p+b+\frac{1}{2}} y^{3a-3p-\frac{3}{2}} e^{-(x+y)} dxdy$$

$$= \frac{2^{3a+b}}{\pi} \int_0^\infty \int_0^\infty x^{b+\frac{1}{2}} y^{-\frac{3}{2}} e^{-(x+y)} (\sum_{p=0}^{a-1} \binom{a}{p} x^{3p} y^{3(a-p)}) dxdy$$

$$= \frac{2^{3a+b}}{\pi} \int_0^\infty \int_0^\infty x^{b+\frac{1}{2}} y^{-\frac{3}{2}} e^{-(x+y)} ((x^3 + y^3)^a - x^{3a}) dxdy$$

The last step follows from the binomial identity $(x^3 + y^3)^a = \sum_{p=0}^{a} \binom{a}{p} x^{3p} y^{3(a-p)}$.

Since $x^3 + y^3 \le (x + y)^3$ holds for nonnegative x, y, we see that

$$(x^3 + y^3)^a - x^{3a} \le (x + y)^{3a} - x^{3a}.$$

Therefore,

$$S \le \frac{2^{3a+b}}{\pi} \int_0^\infty \int_0^\infty x^{b+\frac{1}{2}} y^{-\frac{3}{2}} e^{-(x+y)} ((x + y)^{3a} - x^{3a}) dxdy.$$

We perform a change of variables: let $t = x + y, u = \dfrac{x}{x + y}$, then the Jacobian of this change is t. Then we have

$$S \le \frac{2^{3a+b}}{\pi} \int_0^\infty e^{-t} t^{3a+b} dt \int_0^1 u^{b+\frac{1}{2}} (1 - u)^{-\frac{3}{2}} (1 - u^{3a}) du. \tag{12}$$

By the definition of the beta function and its relationship with the gamma function, we write the inner integral on the right-hand side of (12) as

$$\int_0^1 u^{b+\frac{1}{2}} (1 - u)^{-\frac{3}{2}} (1 - u^{3a}) du = B(b + \frac{3}{2}, -\frac{1}{2}) - B(3a + b + \frac{3}{2}, -\frac{1}{2})$$
$$= 2\sqrt{\pi} (\frac{\Gamma(3a + b + \frac{3}{2})}{\Gamma(3a + b + 1)} - \frac{\Gamma(b + \frac{3}{2})}{\Gamma(b + 1)}).$$

Now, we convert the outer integral of (12) $\int_0^\infty e^{-t} t^{3a+b} dt = \Gamma(3a + b + 1)$; thus, (12) is equivalent to

$$S \le \frac{2^{3a+b+1}}{\sqrt{\pi}} \Gamma(3a + b + 1) \cdot (\frac{\Gamma(3a + b + \frac{3}{2})}{\Gamma(3a + b + 1)} - \frac{\Gamma(b + \frac{3}{2})}{\Gamma(b + 1)})$$

$$\le \frac{2^{3a+b+1}}{\sqrt{\pi}} \Gamma(3a + b + \frac{3}{2}) - \frac{2^{3a+b+1}}{\sqrt{\pi}} \Gamma(3a + b + 1) \frac{\Gamma(b + \frac{3}{2})}{\Gamma(b + 1)} \tag{13}$$

Rearrange (13)

$$\frac{2^{3a+b+1}}{\sqrt{\pi}} \Gamma(3a + b + \frac{3}{2}) - S \ge \frac{2^{3a+b+1}}{\sqrt{\pi}} \Gamma(3a + b + 1) \frac{\Gamma(b + \frac{3}{2})}{\Gamma(b + 1)}.$$

By the relationship between double factorials and the gamma function,

$$\frac{2^{3a+b+1}}{\sqrt{\pi}} \Gamma(3a + b + \frac{3}{2}) = (6a + 2b + 1)!!$$

After substitution, our proof is complete.

$$\begin{aligned}(6a+2b+1)!!-S &\geq \frac{2^{3a+b+1}}{\sqrt{\pi}}\Gamma(3a+b+1)\frac{\Gamma(b+\frac{3}{2})}{\Gamma(b+1)} \\ &\geq 0 \ \text{ for nonnegative } a,b\end{aligned}$$

□

Remark 5.6. The inequality (9) at $d=3a+b$ is equivalent to (11). Substitute $3a+b$ for d, so the left-hand side of (9) is

$$\begin{aligned}&\sum_{p=0}^{a}(-1)^{a-p}\binom{a}{p}(6p+2b+1)!!\prod_{j=1}^{3a-3p}(2b+1+6p-2d+2j)\\ =&\sum_{p=0}^{a}(-1)^{a-p}\binom{a}{p}(6p+2b+1)!!\cdot 1\cdot(-1)\cdots(2b+3+6p-2(3a+b))\\ =&(6a+2b+1)!!-\sum_{p=0}^{a-1}\binom{a}{p}(6p+2b+1)!!(6a-6p-3)!!\end{aligned}$$

5.2. **Proof of Theorem 5.1.** We define forward differences, as they will be used in the proof of Theorem 5.1.

Definition 5.7.

$$(\Delta^a f)(x) := \sum_{k=0}^{a}(-1)^{a-k}\binom{a}{k}f(x+k)$$

is the *a-th order forward difference* of f. The *first order forward difference*, or simply, *forward difference*, is then

$$(\Delta f)(x) = f(x+1)-f(x).$$

We introduce the well-known Leibniz rule for forward differences evaluated at $x=0$:

Lemma 5.8. *Let $a\geq 0$,*

$$\begin{aligned}\Delta^a(uv)(x)\Big|_{x=0} &= \sum_{i=0}^{a}\binom{a}{i}(\Delta^i u)(x+a-i)\cdot(\Delta^{a-i}v)(x)\Big|_{x=0}\\ &= \sum_{i=0}^{a}\binom{a}{i}(\Delta^i u)(a-i)\cdot(\Delta^{a-i}v)(0)\end{aligned}$$

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. (7) is equivalent to the expression

$$(-1)^a\cdot 2^d\cdot\sum_{k=0}^{a}(-1)^k\binom{a}{k}\prod_{r=0}^{d-1}(b+\frac{3}{2}+3k-d+r)\geq 0. \tag{14}$$

Denote

$$p_d(k) := \prod_{r=0}^{d-1}(b+\frac{3}{2}+3k-d+r)$$

Then by the well-known identity

$$\sum_{k=0}^{a}(-1)^k\binom{a}{k}f(k) = (-1)^a(\Delta^a f)(0)$$

where $\Delta^a f$ is the a-th order forward difference of f. We can rewrite (14) as:

$$2^d(\Delta^a p_d)(0) \geq 0. \tag{15}$$

To prove this, proceed by separating $p_d(k)$:

$$p_d(k) = \prod_{r=0}^{d-1}(b+\frac{3}{2}+3k-d+r) = p_{d-1}(k)\cdot(b+\frac{3}{2}+3k-d+(d-1))$$

We denote $q_d(k) := (b+\frac{3}{2}+3k-d+(d-1))$.

Setting $u = q_d$, $v = p_{d-1}$ and plugging into the identity Lemma 5.8, we have

$$\Delta^a(q_d p_{d-1})(0) = \sum_{i=0}^{a}\binom{a}{i}(\Delta^i q_d)(a-i)\cdot(\Delta^{a-i}p_{d-1})(0). \tag{16}$$

Since $q_d(k) = 3k+b+\frac{1}{2}$ is a linear function, $\Delta q_d(k) = q_d(k+1)-q_d(k) = \Delta q_d \equiv 3$ is constant and thus $\Delta^i q_d \equiv 0$ for $i > 1$. We define a recursion function with the only two surviving summands (16):

$$F_d(a) := (\Delta^a p_d)(0) = (\Delta^a q_d p_{d-1})(0) = q_d(a)\cdot F_{d-1}(a) + 3a\cdot F_{d-1}(a-1) \tag{17}$$

Now we prove (14) by induction as follows: first, we note that the LHS of (8) equals zero when $d < a$; since the LHS of (7) is the LHS of (8) multiplied by $a!$, $F_d(a) \equiv 0$ when $d < a$. Also, $q_d(a)$ and $3a$ are both nonnegative for $0 \leq d \leq 3a+b+1$. Next, we see that the base cases $F_0(0) = 1, F_1(1) = \Delta p_1(0) = p_1(1) - p_1(0) = 3$ are positive.

Assume $F_{d-1}(a)$ and $F_{d-1}(a-1)$ are nonnegative for their corresponding bounds $(d-1) \leq 3a+b+1$ and $(d-1) \leq 3(a-1)+b+1$. We examine $F_d(a)$ with bounds $d \leq 3a+b+1$. While the bound for $F_{d-1}(a)$ is automatically satisfied, $F_{d-1}(a-1)$ is not necessarily nonnegative for $d = 3a+b$ and $d = 3a+b+1$; however, we proved those special cases previously.

Thus, by induction, $F_d(a) = (\Delta^a p_d)(0) \geq 0$. The theorem is proved. □

6. Eynard-Orantin topological recursion

Definition 6.1. We call the following generating function

$$F(x_1,\ldots,x_n) = \sum_{g=0}^{\infty}F_g(x_1,\ldots,x_n) = \sum_{g=0}^{\infty}\sum_{\sum d_j=3g-3+n}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g\prod_{j=1}^{n}x_j^{d_j}$$

the n-point function.

Consider the following normalized n-point function

$$G(x_1,\ldots,x_n) = \exp\left(\frac{-\sum_{j=1}^{n}x_j^3}{24}\right)F(x_1,\ldots,x_n).$$

In particular, we have Zagier's formula for 3-point function which we learned from Faber.

$$G(x,y,z) = \sum_{r,s\geq 0}\frac{r!S_r(x,y,z)}{4^r(2r+1)!!\cdot 2}\cdot\frac{\Delta^s}{8^s(r+s+1)!},$$

where $S_r(x,y,z)$ and Δ are the homogeneous symmetric polynomials defined by

$$S_r(x,y,z) = \frac{(xy)^r(x+y)^{r+1} + (yz)^r(y+z)^{r+1} + (zx)^r(z+x)^{r+1}}{x+y+z} \in \mathbb{Z}[x,y,z],$$
$$\Delta(x,y,z) = (x+y)(y+z)(z+x) = \frac{(x+y+z)^3}{3} - \frac{x^3+y^3+z^3}{3}.$$

In Step 1 of the proof of Theorem 4.3, we need to prove that $I_{b,c}(k) \geq 0$ for any $b,c \geq 0$.

On the other hand, it is not difficult to get that

$$I_{b,c}(k) = 24^k k!(2b+1)!!(2c+1)!! \times [x^{3k-b-c}y^bz^c]\left(\exp\left(\frac{y^3+z^3}{24}\right)G(x,y,z)\right),$$

where $[x^{3k-b-c}y^bz^c]$ means taking the corresponding coefficients.

Therefore it suffices to show $[x^ay^bz^c]G_n(x,y,z) \geq 0$ where $G_n(x,y,z)$ is the normalized 3-point function as in Zagier's formula.

We are going to give a proof using the Eynard-Orantin theory [6] which is a powerful and unifying tool for enumerative geometry.

Fix $n \geq 0$ and $a,b,c \geq 0$ with $a+b+c = 3n$. Define generating series

$$J(u) = \sum_{k\geq 0} J_k u^k, \qquad G(u) = \sum_{k\geq 0} G_k u^k, \qquad J(u) = e^{\frac{x^3+y^3+z^3}{24}u}G(u),$$

and use the normalization

$$[x^ay^bz^c]\, J_k = \langle \tau_a\tau_b\tau_c\rangle \quad (\text{if } a+b+c = 3k;\ \text{else } 0).$$

From Cauchy product,

$$[x^ay^bz^c]\, G_n = \sum_{i,j,k\geq 0} \frac{(-1)^{i+j+k}}{24^{\,i+j+k}\, i!\, j!\, k!} \big\langle \tau_{a-3i}\, \tau_{b-3j}\, \tau_{c-3k}\big\rangle, \tag{18}$$

with the convention $\langle \tau_m \cdots\rangle = 0$ if any index is negative.

EO moment lemma. Let $\Delta := \{(s_1,s_2,s_3) \in [0,1]^3 :\ s_1+s_2+s_3 \leq 1\}$. There exists a finite positive Borel measure $\mu_{n;a,b,c}$ on Δ such that, for all $i,j,k \geq 0$,

$$\big\langle \tau_{a-3i}\, \tau_{b-3j}\, \tau_{c-3k}\big\rangle = \int_\Delta s_1^i s_2^j s_3^k \, d\mu_{n;a,b,c}(s_1,s_2,s_3). \tag{19}$$

Proof of the lemma (via Airy EO in one page). Work on the Airy curve $x = \frac{1}{2}z^2,\ y = z$ with kernel

$$K(z_0,z) = \frac{dz_0}{2z^2}\Big(\frac{1}{z_0+z} - \frac{1}{z_0-z}\Big),$$

and EO recursion

$$W_{g,n+1}(z_0,z_S) = \operatorname*{Res}_{z\to 0} K(z_0,z)\Big[W_{g-1,n+2}(z,-z,z_S) + \sum_{\substack{g_1+g_2=g\\ S=I\sqcup J}} W_{g_1,|I|+1}(z,z_I)\, W_{g_2,|J|+1}(-z,z_J)\Big].$$

The coefficient dictionary is

$$W_{g,3}(z_1,z_2,z_3) = \sum_{a+b+c=3g} \langle \tau_a\tau_b\tau_c\rangle \prod_{r=1}^{3} \frac{(2d_r-1)!!}{z_r^{2d_r+2}}\, dz_r.$$

Apply the inverse Laplace transform variablewise:

$$\mathcal{L}^{-1}_{z\to L}\Big(\frac{dz}{z^{2m+2}}\Big) = \frac{L^{2m+1}}{(2m+1)!}\, dL, \qquad \mathcal{L}^{-1}_{z\to L}\Big((z+w)^{-m-1}\Big) = \frac{L^m}{m!}e^{-wL}\, dL.$$

Each EO term becomes an integral over parameters with *nonnegative* polynomial kernels:

(1) Pairing node $\Rightarrow$ parameter $s \in [0,1]$ and Beta kernel

$$B^{(\alpha,\beta)}(s) := \frac{(\alpha+\beta+1)!}{\alpha!\,\beta!}\, s^{\alpha}(1-s)^{\beta}, \quad \alpha = 1,\ \beta = 2d+1.$$

It replaces the first length by $L_1' = sL_1 + (1-s)L_j$.

(2) Two-leg node $\Rightarrow$ parameters (s,t) with $s,t \geq 0$, $s+t \leq 1$ and triangle-Beta kernel

$$C^{(\alpha,\beta,\gamma)}(s,t) := \frac{(\alpha+\beta+\gamma+2)!}{\alpha!\,\beta!\,\gamma!}\, s^{\alpha}t^{\beta}(1-s-t)^{\gamma}, \quad \alpha = 2d_1{+}1,\ \beta = 2d_2{+}1,\ \gamma = 1,$$

replacing the first length by $L_1' = (s+t)L_1$.

Unroll the recursion into finitely many trees T with parameter polytopes Ω_T (products of $[0,1]$ and $\{s,t \geq 0,\ s{+}t \leq 1\}$). Compose the affine updates to obtain $\pi_T : \Omega_T \to \Delta$, $\theta \mapsto (s_1(\theta), s_2(\theta), s_3(\theta))$. Let $K_T(\theta)$ be the product of the corresponding Beta polynomials, and let $C_T^{(a,b,c)} \geq 0$ be the (finite) sum of multinomial/normalization/splitting constants selecting the root power $L_1^{2a}L_2^{2b}L_3^{2c}$. Define

$$\mu_{n;a,b,c} := \sum_T C_T^{(a,b,c)}\,(\pi_T)_\#\big(K_T(\theta)\,d\theta\big).$$

Then coefficient extraction shows that each drop $a \mapsto a-3$ (resp. $b \mapsto b-3$, $c \mapsto c-3$) multiplies the integrand by $s_1(\theta)$ (resp. $s_2(\theta)$, $s_3(\theta)$), while K_T and $C_T^{(a,b,c)}$ are independent of (i,j,k) and nonnegative. This yields (19). □

Conclusion. Insert (19) into (18) and sum the (separate) exponential series:

$$\begin{aligned}[x^a y^b z^c]\,G_n &= \int_\Delta \Big(\sum_{i\geq 0}\frac{(-s_1/24)^i}{i!}\Big)\Big(\sum_{j\geq 0}\frac{(-s_2/24)^j}{j!}\Big)\Big(\sum_{k\geq 0}\frac{(-s_3/24)^k}{k!}\Big)\, d\mu_{n;a,b,c} \\ &= \int_\Delta e^{-(s_1+s_2+s_3)/24}\, d\mu_{n;a,b,c}.\end{aligned}$$

The integrand and measure are nonnegative, hence $[x^a y^b z^c]\,G_n \geq 0$.

7. Examples

Following the method used by Afandi [1], we give some examples of Ehrhart polynomials $L_{\mathcal{P}}(g)$ and their explicit partial polytopal complexes. We follow the definition of triangulations and introduce a theorem involving them as given in [1].

Definition 7.1. Let $\mathcal{K}$ be a partial polytopal complex of dimension d. A *triangulation* $\mathcal{T}$ of $\mathcal{K}$ is a disjoint union of open simplices whose support is $\mathcal{K}$. The triangulation $\mathcal{T}$ is *unimodular* if the closure of each open simplex in $\mathcal{T}$ is lattice equivalent to the standard simplex. The f^*-vector of $\mathcal{T}$, $(f_0^*, \dots f_d^*)$, has a special meaning. In particular, $f_i^* = \#\{i\text{-dimensional open simplices in } \mathcal{T}\}$.

Theorem 7.2. *Let $\mathcal{K}$ be a partial polytopal complex of dimension d and let $\mathcal{T}$ be a unimodular triangulation of $\mathcal{K}$. Then*

$$L_{\mathcal{K}}(g) = \sum_{i=0}^{d} f_i^* \binom{g-1}{i}$$

where $(f_0^*, \dots, f_d^*)$ is the f^*-vector of $\mathcal{T}$.

The polytopes we give as examples will be inside-out polytopes, they are defined as the following:

Definition 7.3. An *inside-out polytope* is any set of the form

$$\mathcal{P} \setminus \left(\bigcup_{H \in \mathcal{H}} H \right)$$

where $\mathcal{P} \subseteq \mathbb{R}^d$ is a full dimensional integral d-polytope, and let $\mathcal{H}$ is a *hyperplane arrangement*, which is, a finite collection of hyperplanes in $\mathbb{R}^d$.

Note that although we will give an example of a polytope that matches its corresponding integer-valued polynomial perfectly, each polynomial is not exclusively related to one polytope. This will be illustrated through a secondary (and reduced) polytope presented in each example; the Ehrhart polynomial of this polytope will differ from that of the primary example by a factor.

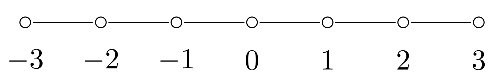
7.1. **Polynomial** $P_{1,0}(g-1)$**.** We show that the integer-valued polynomial

$$P_{1,0}(g-1) = 6g - 6 = 6\binom{g-1}{1}$$

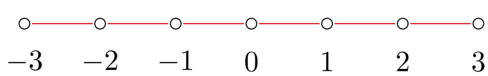
corresponds to the Ehrhart polynomial

$$L_{\mathcal{P}_{1,0}}(g) = |g\mathcal{P}_{1,0} \cap \mathbb{Z}|.$$

Define $\mathcal{P}_{1,0}$ as the inside-out polytope $[-3,3] \setminus \{\pm 3, \pm 2, \pm 1, 0\}$, shown below.

$\mathcal{P}_{1,0} =$ 

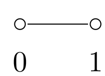
$\mathcal{P}_{1,0}$ has unimodular triangulation $\mathcal{T}_{1,0}$:

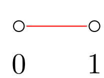
$\mathcal{T}_{1,0} =$ 

We see that $\mathcal{T}_{1,0}$ has f^*-vector $(0,6)$, so by Theorem 7.2,

$$L_{\mathcal{P}_{1,0}}(g) = 6\binom{g-1}{1} = P_{1,0}(g-1).$$

Alternatively, we can consider the reduced inside-out polytope $\widetilde{\mathcal{P}_{1,0}} := [0,1] \setminus \{0,1\}$ and its unimodular triangulation $\widetilde{\mathcal{T}_{1,0}}$:

$\widetilde{\mathcal{P}_{1,0}} =$ 

$\widetilde{\mathcal{T}_{1,0}} =$ 

Since $\widetilde{\mathcal{T}_{1,0}}$ has $(f_0^*, f_1^*) = (0,1)$, by Theorem 7.2,

$$L_{\widetilde{\mathcal{P}_{1,0}}}(g) = 1\binom{g-1}{1}.$$

Hence we have

$$P_{1,0}(g-1) = 6 \cdot L_{\widetilde{\mathcal{P}_{1,0}}}(g) = 6|g\widetilde{\mathcal{P}_{1,0}} \cap \mathbb{Z}|.$$

7.2. **Polynomial** $P_{1,1}(g-1)$**.** Now we show that

$$P_{1,1}(g-1) = 36(g-1)^2 - 18(g-1) = 36g^2 - 90g + 54 = 18\binom{g-1}{1} + 72\binom{g-1}{2}$$

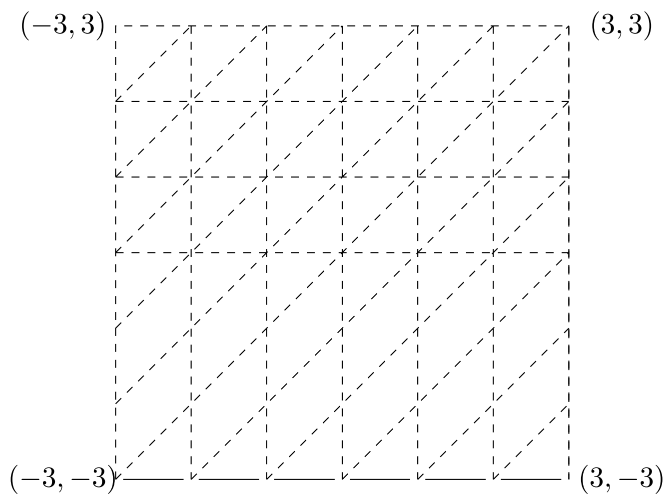
corresponds to the Ehrhart polynomial

$$L_{\mathcal{P}_{1,1}}(g) = |g\mathcal{P}_{1,1} \cap \mathbb{Z}^2|,$$

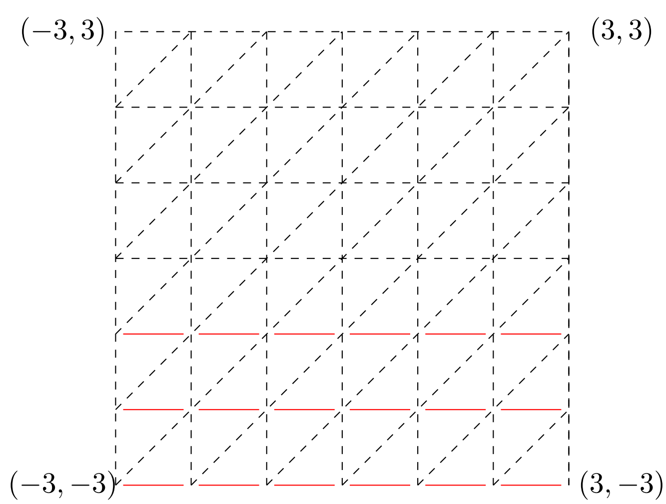
where $\mathcal{P}_{1,1}$ represents the inside-out polytope $([-3,3]\times[-3,3]) \setminus \mathcal{H}$, and $\mathcal{H}$ is the hyperplane arrangement

$$\begin{aligned}\mathcal{H} := \{&x_2 = 0, x_2 = 1, x_2 = 2, x_2 = 3, x_2 = x_1,\\ &x_1 = 0, x_1 = \pm 1, x_1 = \pm 2, x_1 = \pm 3,\\ &x_2 = x_1 \pm 1, x_2 = x_1 \pm 2, x_2 = x_1 \pm 3, x_2 = x_1 \pm 4, x_2 = x_1 \pm 5\}.\end{aligned}$$

A visualization of $\mathcal{P}_{1,1}$ is shown below.



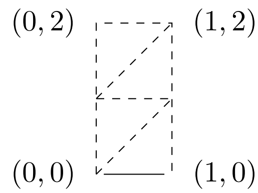
$\mathcal{P}_{1,1}$ admits the following unimodular triangulation:



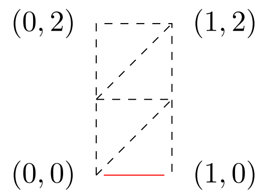
The f^*-vector of the above triangulation is $(0, 18, 72)$. Thus, by Theorem 7.2,

$$L_{\mathcal{P}_{1,1}}(g) = 0\binom{g-1}{0} + 18\binom{g-1}{1} + 72\binom{g-1}{2} = P_{1,1}(g-1).$$

On the other hand, consider the simplified inside-out polytope $\widetilde{\mathcal{P}_{1,1}} = ([0,1] \times [0,2]) \setminus \mathcal{H}'$, where $\mathcal{H}' = \{x_2 = 1, x_2 = 2, x_1 = 0, x_2 = x_1, x_2 = x_1 + 1\}$. Below is a visualization of $\widetilde{\mathcal{P}_{1,1}}$.



$\widetilde{\mathcal{P}_{1,1}}$ admits the unimodular triangulation with f^*-vector $(0, 1, 4)$.



By Theorem 7.2, $L_{\widetilde{\mathcal{P}_{1,1}}}(g) = 0\binom{g-1}{0} + 1\binom{g-1}{1} + 4\binom{g-1}{2}$ and we have

$$P_{1,1}(g-1) = 18 \cdot L_{\widetilde{\mathcal{P}_{1,1}}}(g) = 18|g\widetilde{\mathcal{P}_{1,1}} \cap \mathbb{Z}^2|.$$

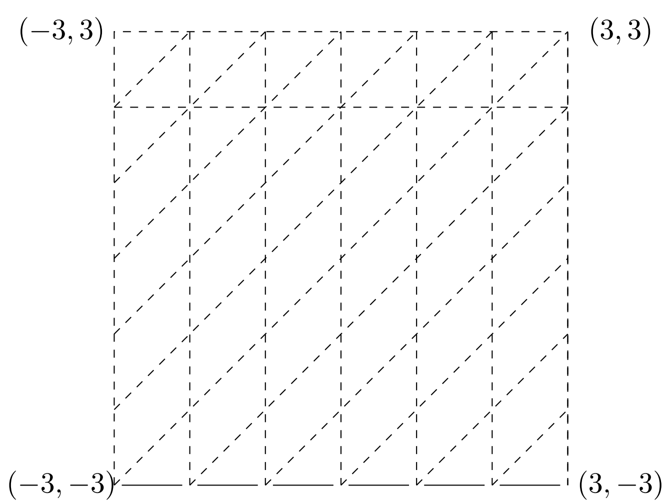
7.3. **Polynomial** $P_{2,0}(g-1)$**.** Finally, we show that $P_{2,0}(g-1) = 30\binom{g-1}{1} + 72\binom{g-1}{2}$ corresponds to the Ehrhart polynomial

$$L_{\mathcal{P}_{2,0}}(g) = |g\mathcal{P}_{2,0} \cap \mathbb{Z}^2|,$$

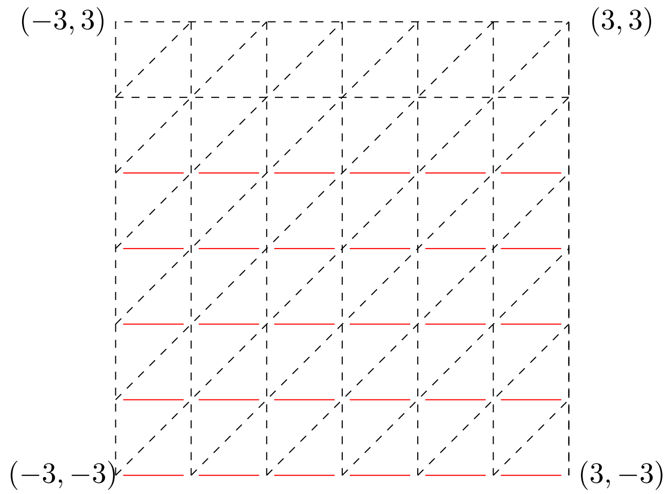
where $\mathcal{P}_{2,0}$ is the inside-out polytope $([-3,3] \times [-3,3]) \setminus \mathcal{H}_2$, and $\mathcal{H}_2$ is the hyperplane arrangement

$$\mathcal{H}_2 := \{x_2 = 2, x_2 = 3, x_2 = x_1, x_1 = 0, x_1 = \pm 1, x_1 = \pm 2, x_1 = \pm 3,$$
$$x_2 = x_1 \pm 1, x_2 = x_1 \pm 2, x_2 = x_1 \pm 3, x_2 = x_1 \pm 4, x_2 = x_1 \pm 5\}.$$

Here is a visualization of $\mathcal{P}_{2,0}$:



$\mathcal{P}_{2,0}$ admits the following unimodular triangulation $\mathcal{T}_{2,0}$.



$\mathcal{T}_{2,0}$ has f^*-vector $(0, 30, 72)$, and by Theorem 7.2, we see that

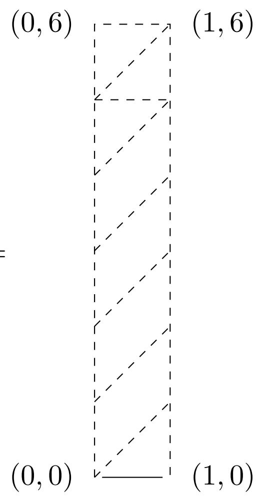
$$L_{\mathcal{P}_{2,0}}(g) = 0\binom{g-1}{0} + 30\binom{g-1}{1} + 72\binom{g-1}{2} = P_{2,0}(g-1).$$

Alternatively, consider the inside-out polytope $\widetilde{\mathcal{P}_{2,0}} := ([0,1] \times [0,6]) \setminus \mathcal{H}'_2$ where $\mathcal{H}'_2$ is the hyperplane arrangement

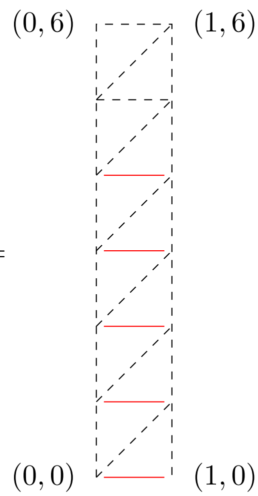
$$\mathcal{H}'_2 := \{x_1 = 0, x_1 = 1, x_2 = 5, x_2 = 6, x_2 = x_1, x_2 = x_1+1, x_2 = x_1+2, x_2 = x_1+3, x_2 = x_1+4, x_2 = x_1+5\}.$$

Below is a visualization of $\widetilde{\mathcal{P}_{2,0}}$ and its unimodular triangulation $\widetilde{\mathcal{T}_{2,0}}$:

$\widetilde{\mathcal{P}_{2,0}} =$



$\widetilde{\mathcal{T}_{2,0}} =$



$\widetilde{\mathcal{T}_{2,0}}$ has f^*-vector $(0, 5, 12)$. Thus, by Theorem 7.2, $L_{\widetilde{\mathcal{P}_{2,0}}}(g) = 0\binom{g-1}{0} + 5\binom{g-1}{1} + 12\binom{g-1}{2}$. Hence,

$$P_{2,0}(g-1) = 6 \cdot L_{\widetilde{\mathcal{P}_{2,0}}}(g) = 6|g\widetilde{\mathcal{P}_{2,0}} \cap \mathbb{Z}^2|.$$

8. Observations when proving Theorem 5.1

When checking Theorem 5.1 computationally, we found that for any a, b values tested, the left-hand side of (7), i.e. the expression

$$\sum_{p=0}^{a}(-1)^{a-p}\binom{a}{p}\prod_{j=1}^{d}(2b+3+6p-2j), \tag{20}$$

becomes drastically negative when we set $d = 3a + b + 2$. Below is a table comparing the value of the above expression when $d = 3a + b + 2$ against its value when $d = 3a + b + 1$, for certain a, b.

a	b	(20) when $d = 3a + b + 1$	(20) when $d = 3a + b + 2$
1	4	2041200	−2126250
2	3	659874600	−697296600
3	5	6205946712966000	−6310673040672000
4	6	8212934528414231616000	−8290585205214071760000

Since the LHS of (9),

$$\sum_{p=0}^{a}(-1)^{a-p}\binom{a}{p}(6p+2b+1)!!\prod_{j=1}^{3a-3p}(2b+1+6p-2d+2j), \tag{21}$$

is a polynomial in d, it is possible to visualize it as a function of d for fixed a, b in a graph. For instance, below is the graph of the polynomial in d when $a = b = 5$:

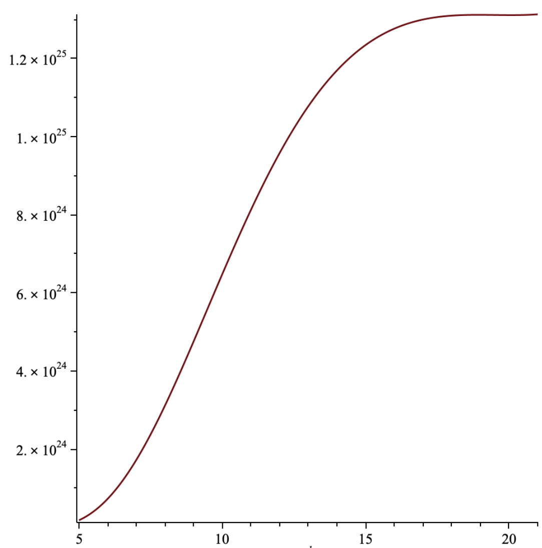


Figure 1. (21) when $a = b = 5$, drawn with Maple

Although it may seem like it, the derivative of the above function is not always positive; at around $d = 19$ to $d = 20$, the derivative briefly drops below zero. Here is a graph of the derivative of (21) when $a = b = 5$.

Remarkably, the minimum value of the first derivative of (21) stays around two orders of magnitude less than the maximum value, independent of a, b. We denote by $l_{x,y}$ the length of

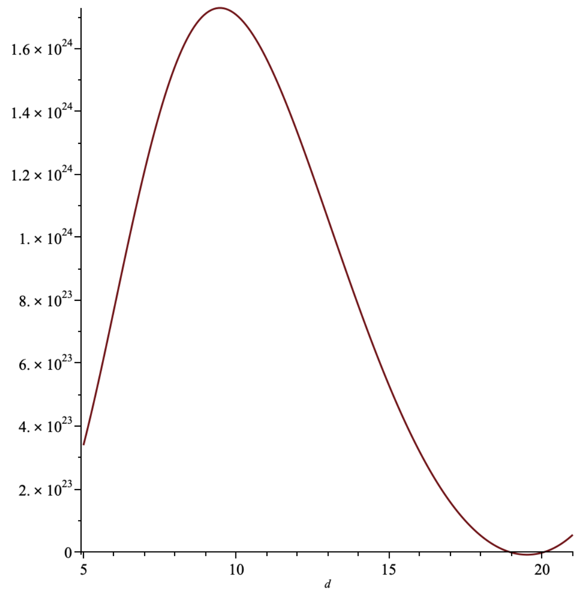


FIGURE 2. First derivative of (21) when $a = b = 5$, drawn with Maple

the interval where the derivative of (21) (when $a = x, b = y$) is negative. As a, b increase, $l_{a,b}$ shrinks. For example, $l_{5,5}$ is approximately equal to 1.15620317, but when $l_{6,6} \approx 1.15552147$. As a, b become large, $l_{a,b}$ seems to exhibit asymptotic behavior: $l_{50,50} \approx 1.1547080$, whereas $l_{100,100} \approx 1.1547024$ – a significantly smaller decrease in length than before despite the large difference in values of a, b.

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Declaration of Academic Integrity

The participating team declares that the paper submitted is comprised of original research and results obtained under the guidance of the instructor. To the team's best knowledge, the paper does not contain research results, published or not, from a person who is not a team member, except for the content listed in the references and the acknowledgment. If there is any misinformation, we are willing to take all the related responsibilities.

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